Physics 216, Final Exam

December 16, 2013, 2 hours

Solve ten problems out of twelve

1. Check Stokes' theorem using the function $\overrightarrow{V} = ay\widehat{x} + bx\widehat{y}$ (a and b are constants) and the circular path of radius R, centered at the origin in the $x-y$ plane.

2. Show that:

   (a) If A is a diagonal matrix with all diagonal elements different, and B is a matrix commuting with A ($AB = BA$) then B is diagonal.

   (b) The trace of a product of a symmetric and antisymmetric matrix is zero.

3. Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

4. Use the exponential Fourier series to evaluate the expansion of the delta function $\delta(\varphi_1 - \varphi_2)$. Use the result to verify that

$$\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\pi}^{\pi} f(\varphi_1)\, e^{im(\varphi_1-\varphi_2)} d\varphi_1 = f(\varphi_2)$$

5. A rectangular pulse is described $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$. Find the exponential Fourier transform $F(t)$ then use Parseval's theorem to evaluate the integral $\int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2} dt$.

6. Let $\{Q_n, \qquad n = 0, \cdots, \infty\}$ be an orthogonal set of Sturm-Liouville polynomials with respect to the weight function $w(x)$

$$\int_{-\infty}^{\infty} w(x)\, Q_n(x)\, Q_m(x)\, dx = 0, \qquad n \neq m$$

where $Q_n\ (x)$ satisfies the differential equation $\frac{1}{w}\frac{d}{dx}\left(w\alpha(x)\frac{dQ_n}{dx}\right) + \lambda_n Q_n = 0$. Show that $\left\{Q_n' = \frac{dQ_n}{dx}, \qquad n = 1, \cdots, \infty\right\}$ satisfy the property

$$\int_{-\infty}^{\infty} w(x)\, \alpha(x)\, Q_n'(x)\, Q_m'(x)\, dx = 0, \qquad n \neq m$$

Hint: Integrate by parts only on one of the $\frac{dQ_n}{dx}$ and use the defining equation for Q_n.

7. Using only the generating function

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

show that $J_n(-x) = (-1)^n J_n(x)$.

8. The quantum mechanical angular momentum operator is given by $\overrightarrow{L} = i\overrightarrow{r} \times \overrightarrow{\nabla}$. Show that

$$L_{\pm} = L_x \pm iL_y = \pm e^{\pm i\varphi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\varphi}\right)$$

9. Consider the Hermite differential operator $2H = -\frac{d^2}{dx^2} + x^2$ and the raising and lowering operators $a_{\pm} = \frac{1}{\sqrt{2}}\left(x \mp \frac{d}{dx}\right)$. Show that $[a_-, a_+] = 1$, and $[N, a_{\pm}] = \pm a_{\pm}$ where $N = a_+ a_-$. Express H in terms of N. Let $\psi_n(x)$ be the eigenstate satisfying $N\psi_n(x) = n\psi_n(x)$. Deduce that $\psi_n(x)$ is an eigenstate of H with eigenvalue $\left(n + \frac{1}{2}\right)$, i.e. $H\psi_n(x) = \left(n + \frac{1}{2}\right)\psi_n(x)$.

10. Find the Laurent series expansion of the function $f(z) = \frac{1}{(z-1)(z-2)}$ in the annulus $1 < |z| < 2$ by direct expansion. (Hint: first express $f(z)$ in terms of $\frac{1}{z-1}$ and $\frac{1}{z-2}$ and use power series expansion in region of validity). Verify the coefficients a_0 and a_{-1} using the formula $a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}dz$ where $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$.

11. Find the residue at $z = -i$ of the function $\frac{1}{z^4-1}$. Find the integral $\oint_C \frac{1}{z^4-1}dz$ where C is a circle of radius $\frac{1}{2}$ centered at $z = -i$.

12. Using contour integration evaluate the definite integral $\int_0^{2\pi} \frac{1}{3+2\cos\theta}d\theta$.